Supplemental Notes

EE503 Week 08 Dr. Franzke

HW #8

1. Gubner Ch. 14, 14.1 - 14.3
2. Leon Garcia 7.15 - 7.17, 7.41, 7.44 - 7.45, 7.50

Topics

1. Key acronyms:

   uc MOPeD, $m \to P \to D$, (ueOPD)

   MI, CI, WLLN (SLLN)

2. Stochastic convergence

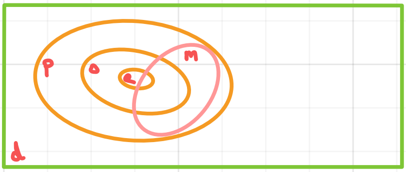
   $a_n \xrightarrow{e} a$ iff $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{Z}^+ . \; \forall n \geq n_0 \; |a_n - a| < \varepsilon$

   trick: pick a_n and a to give e, m, p, d

	a_n	a	
e - everywhere	$X_n(w)$	$X(w)$	
m - mean square	$E[(X_n - X)^2]$	0	
p - in probability	$P[\lvert X_n - X \rvert > \varepsilon]$	0	$\forall \varepsilon > 0$
d - in distribution	$F_{X_n}(x)$ at points of continuity x	$F_X(x)$	

o - with probability one : $P\left(\lim_{n \to \infty} X_n = X \right)$

$m \longrightarrow p \longrightarrow d$



Acronym: UC MOPED ☆☆

U: uniform convergence

$$X_n \xrightarrow{u} X \text{ iff } \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 \ |X_n(\omega) - X(\omega)| < \varepsilon \ \forall x.$$

$\exists \forall$

C: Cauchy criterion

$$X_n \xrightarrow{c} X \text{ iff} : \forall \omega : \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 \ \forall m \geq n_0$$

(since $\mathbb{R}$ is "complete")

$$|X_n(\omega) - X_m(\omega)| < \varepsilon$$

M: Mean-square convergence

$$X_n \xrightarrow{m} X \text{ iff } \lim_{n \to \infty} E[(X_n - X)^2] = 0$$

Cauchy Criterion for MS convergence

$$\text{iff } \lim_{n \to \infty} \lim_{m \to \infty} E[(X_n - X_m)^2] = 0$$

O: Probability One ("strong" "almost surely") convergence

$$X_n \xrightarrow{o} X \text{ iff } P[\lim_{n \to \infty} X_n = X] = 1 \text{ iff } P[\lim_{n \to \infty} X_n \neq X] = 0$$

P: Convergence in Probability ("weak")

$$X_n \xrightarrow{P} X \text{ iff } \forall \varepsilon > 0 \ \lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0$$

$$\text{iff } \lim_{n \to \infty} P(|X_n - X| \leq \varepsilon) = 1$$

E: Convergence Everywhere

$$X_n \xrightarrow{e} X \text{ iff } \forall w \in \Omega \ \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{Z}^+ \ \forall n \ge n_0$$

$\forall \exists$

$$|X_n(w) - X(w)| < \varepsilon$$

D: Convergence in Distribution ("in law")

$$X_n \xrightarrow{d} X \text{ iff } \lim_{n\to\infty} F_{X_n}(x) = F_X(x) \text{ at points of continuity}$$

Continuity Theorem: If g is continuous

$$\left\{\begin{array}{l} X_n \xrightarrow{o} X \\ X_n \xrightarrow{p} X \\ X_n \xrightarrow{d} X \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} g(X_n) \xrightarrow{o} g(X) \\ g(X_n) \xrightarrow{p} g(X) \\ g(X_n) \xrightarrow{d} g(X) \end{array}\right\}$$

Note: $X_n \xrightarrow{m} X \not\Longrightarrow g(X_n) \xrightarrow{m} g(X)$

$$\boxed{m \longrightarrow p \longrightarrow d.}$$ ☆

"Mom Paid Dad" (mnemonic)

- Often easier to prove m (or p) than p (or d)

$$u \longrightarrow e \longrightarrow o \longrightarrow p \longrightarrow d.$$

"Uncle Ed Once Paid Dad"

- U implies all, even m.

Note: $e \not\longrightarrow m$

"Random Sampling" (r.s.)

$\longleftrightarrow$ iid $X_1, X_2, \ldots$ (and usually $\sigma_X^2 < \infty$ unless $X \sim$ Cauchy)

☆ **Thrm:** ("Sampling Statistics") If r.s. iid $X_1 \ldots X_n$ and $\sigma_X^2 < \infty$

(1) $E[\bar{X}_n] = \mu_X$

(2) $V[\bar{X}_n] = \dfrac{\sigma_X^2}{n}$ (false if $X_k \sim$ Cauchy)

If randomly sample (of size n) from a finite population (of size N) without replacement:

(1) still holds, but (2) becomes (3) $V[\bar{X}_n] = \frac{\sigma_x^2}{n} \cdot \underbrace{\frac{N-n}{N-1}}_{\text{correction factor}}$

2 Main Limit Theorems

$\frac{1}{n}$ ① Law of large numbers (LLN) (r.s., $\sigma_x^2 < \infty$)

$$\bar{X}_n \xrightarrow{mopd} \mu_x$$

$\frac{1}{\sqrt{n}}$ ② Central limit theorem (CLT)

$$Z_n \xrightarrow{d} Z \sim N(0,1)$$

for $Z_n = Std(\bar{X}_n) = \frac{\bar{X}_n - \mu_x}{\sigma_x / \sqrt{n}} = \frac{\sum_{k=1}^{n} X_k - n \cdot \mu_x}{\sqrt{n} \cdot \sigma_x}$

Defn: Bias B of r.v. estimator $\hat{\theta}_n$

$$B = \left| E[\hat{\theta}_n] - \theta \right|$$

Defn: $\hat{\theta}_n$ is an unbiased estimator (u.B.) of θ iff

$$E[\hat{\theta}_n] = \theta \quad \forall n.$$

Defn: $\hat{\theta}_n$ is asymptotically unbiased for θ iff.

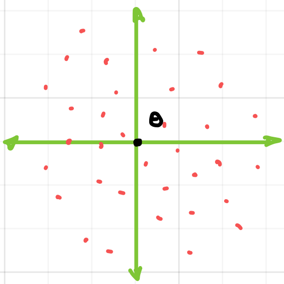
$$\lim_{n\to\infty} E[\hat{\theta}_n] = \theta$$

Thrm: ("Variance-Bias Decomposition")

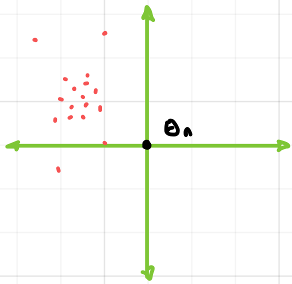
$$MSE[\hat{\theta}_n] = V[\hat{\theta}_n] + (E[\hat{\theta}_n] - \theta)^2$$

for $MSE[\hat{\theta}_n] = E[(\hat{\theta}_n - \theta_n)^2]$.

V vs B tradeoff



V-high, B-low



V-low, B-high.

Prf: $MSE[\hat{\theta}_n] = E[(\theta_n - \theta)^2]$

$$= E[((\theta_n - E[\theta_n]) + (E[\theta_n] - \theta))^2].$$

$$= \underbrace{E[(\hat{\theta}_n - E[\hat{\theta}_n])^2]}_{V[\hat{\theta}_n]} + \underbrace{(E[\theta_n] - \theta)^2}_{B^2}$$

$$\cancel{+ 2(E[\hat{\theta}_n - \theta] - \theta) \cdot \underbrace{E[\hat{\theta}_n - E[\hat{\theta}_n]]}_{E[\hat{\theta}_n] - E[\hat{\theta}_n] = 0}}.$$

$$= V[\hat{\Theta}_n] + \text{Bias}^2[\hat{\Theta}_n].$$

QED

Always ask: Is $\hat{\Theta}_n$ unbiased?
(if not is it asymptotically unbiased)

Ex: $\underbrace{\overline{X}_n}_{(\text{r.s. and } \sigma_x^2)}$ is unbiased for μ_x by theorem (1) $\underbrace{\text{above}}_{E[\overline{X}_n] = \mu_x}$

Thrm: If iid $X_1, \dots, X_n$ and $E[|X|^4] < \infty$ then
($\therefore V[S^2]$ exists)

S_n^2 is unbiased for σ_x^2: $E[S_n^2] = \sigma_x^2$.

Prf:
$$S_n^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2$$
$$= \frac{1}{n-1} \sum_{k=1}^{n} (X_k^2 + \overline{X}_n^2 - 2X_k\overline{X}_n)$$
$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 + n\cdot\overline{X}_n^2 - 2\overline{X}_n \sum_{k=1}^{n} X_k \right)$$
$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 + n\cdot\overline{X}_n^2 - 2n\overline{X}_n \underbrace{\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right)}_{\overline{X}_n} \right)$$
$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 + n\cdot\overline{X}_n^2 - 2n\cdot\overline{X}_n^2 \right)$$
$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 - n\cdot\overline{X}_n^2 \right)$$
$$\therefore E[S_n^2] \overset{\text{linearity}}{=} \frac{1}{n-1} \left(\sum_{k=1}^{n} E[X_k^2] - n\cdot E[\overline{X}_n^2] \right)$$
$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} (V[X_k] + E^2[X_k]) - n\cdot(V[\overline{X}_n] + E^2[\overline{X}_n]) \right)$$

since $V[X] = E[X^2] - E^2[X]$ and finite 4^{th} moment $E[X^4]$.

$$= \frac{1}{n-1}\left(\sum_{k=1}^{n} (\sigma_x^2 + \mu_x^2) - n \cdot \left(\frac{\sigma_x^2}{n} + \mu_x^2 \right) \right)$$

$$= \frac{1}{n-1} \left(n \cdot \sigma_x^2 + \cancel{n \cdot \mu_x^2} - \sigma_x^2 - \cancel{n \cdot \mu_x^2} \right)$$

$$= \frac{n\sigma_x^2 - \sigma_x^2}{n-1}$$

$$= \sigma_x^2 \; \cancel{\frac{n-1}{n-1}}$$

$$= \sigma_x^2$$

QED

Population median m for $X \sim f(x)$:

$$P[X \leq m] \leq \frac{1}{2} \qquad P[X \geq m] \geq \frac{1}{2}$$

Sample median M_n (r.s.)

$$M_n = \text{Median}(X_1, \ldots, X_n)$$

Q: Is M_n U.B. for μ_x?

A: Sometimes

Thrm: If
- iid $X_1, \dots, X_n \sim f(x)$
- $f(x)$ is symmetric about μ_x:

$$E[M_n] = \mu_x \quad \text{(U.B.)}$$

Prf:
$$E[M_n] - \mu_x = E[\text{Median}(X_1, \dots, X_n)] - \mu_x$$

$$= E[\text{Median}(X_1 - \mu_x, \dots, X_n - \mu_x)].$$

additive constant does not affect order

$$= E[\text{Median}(-(X_1 - \mu_x), \dots, -(X_n - \mu_x))].$$

since f symmetric about μ_x

$$= -E[\text{Median}(X_1 - \mu_x, \dots, X_n - \mu_x)].$$

since -1 reverses order

$$= -E[\text{Median}(X_1, \dots, X_n) - \mu_x].$$

$$= -E[M_n] + \mu_x$$

$$\therefore 2E[M_n] = 2\mu_x$$

$$\therefore E[M_n] = \mu_x \quad \text{(U.B.)}$$

QED

★ **Thrm:** ("Markov Inequality", MI) If $X \geq 0$ and $c \in \mathbb{R}^+$ and $E[X]$ exists $P[X \geq c] \leq \frac{E_x[X]}{c}$

Prf: (continuous case)

$$E_x[X] = \int_0^\infty x \cdot f_x(x)\, dx$$

$$= \int_0^c \underbrace{x \cdot f_x(x)\,dx}_{\geq 0} + \int_c^\infty x \cdot f_x(x)\,dx$$

since $x \geq 0$ and f: pdf

$$\geq \int_c^\infty x \cdot f_x(x)\,dx. \qquad \text{since } x \geq 0$$

$$= \int_{x \geq c} x \cdot f_x(x)\,dx \geq \int_{x \geq c} c \cdot f_x(x)\,dx$$

$$= c \cdot \int_c^\infty f_x(x)\,dx = c \cdot P[X \geq c]$$

$$\therefore P[X \geq c] \leq \frac{E_x[X]}{c}$$

QED

Thrm: ("Chebyshev Inequality", CI) If $\sigma_x^2 < \infty$:

for $\forall \varepsilon > 0$: $P[|X - \mu_x| > \varepsilon] \leq \dfrac{V_x[X]}{\varepsilon^2}$

Prf: Pick $\varepsilon > 0$:

$$P[|X-\mu| > \varepsilon] = P[(X-\mu)^2 > \varepsilon^2]$$

$$\overset{\text{M.I.}}{\leq} \frac{E_x[(X-\mu)^2]}{\varepsilon^2} = \frac{V_x[X]}{\varepsilon^2}$$

QED

Put $\varepsilon = k\sigma$ for $k \in \mathbb{Z}^+$ $\quad \therefore P[|X-\mu| > k\sigma] \leq \frac{1}{k^2}$

$$\therefore 1 - P[|X-\mu_x| > k\sigma] \geq 1 - \frac{1}{k^2}$$

$$\therefore \boxed{P[|X-\mu_x| \leq k\sigma^2] \geq 1 - \frac{1}{k^2}}$$

$\therefore P[X \text{ within } \pm 3\sigma \text{ of mean}] \geq \frac{8}{9}$ $\quad (\sigma^2 < \infty)$

★

Thrm: ("Weak" law of large numbers, WLLN) If i.i.d.

$X_1, X_2, \ldots$ & $\sigma_X^2 < \infty$ (r.s.) $\bar{X}_n \xrightarrow{P} \mu_X$.

Pf: $\forall \varepsilon > 0$:

$$\lim_{n\to\infty} P\left[|\bar{X}_n - \mu_X| > \varepsilon\right] \overset{\text{thrm}}{=} \lim_{n\to\infty} P\left[|\bar{X}_n - E[\bar{X}_n]| > \varepsilon\right].$$

$$\overset{\text{C.I.}}{\leq} \lim_{n\to\infty} \frac{V[\bar{X}_n]}{\varepsilon^2}$$

$$= \lim_{n\to\infty} \frac{\sigma_X^2}{n \cdot \varepsilon^2}$$

$$= 0$$

QED

Thrm: ("Strong" LLN, SLLN) $\bar{X}_n \xrightarrow{a} \mu_X$ if r.s.

Thrm: ("Mean-square" LLN, MSLLN) $\bar{X}_n \xrightarrow{m} \mu$ if r.s

Caveat: No such thing as a "law of averages"

Thrm: ("Consistency") $\hat{\theta}_n$ is consistent for θ iff $\hat{\theta}_n \xrightarrow{P} \theta$

($\therefore$ $\bar{X}_n$ is consistent for μ_X)

Thrm: $X_n \xrightarrow{m} X \longrightarrow X_n \xrightarrow{P} X$

Pf: Say $X_n \xrightarrow{m} X$: $\lim_{n\to\infty} E[(X_n - X)^2] = 0$. Pick $\varepsilon > 0$.

$$\lim_{n\to\infty} P[|X_n - X| > \varepsilon] = \lim_{n\to\infty} P[(X_n - X)^2 > \varepsilon^2]$$

$$\overset{M.I.}{\leq} \lim_{n\to\infty} \frac{E[(X_n - X)^2]}{\varepsilon^2}$$

$$= 0 \qquad \text{(by hypothesis)}$$

QED.

∴ Issue spotting sequence for m-p-d problems:

Use $MS = V + B^2$ and $m \longrightarrow p \longrightarrow d$

Q1: Is $\hat{\theta}_n$ unbiased (U.B.)? Asymptotically U.B.?

Q2: Is $\hat{\theta}_n$ consistent? (does $V \downarrow 0$)

If Q1 and Q2 yes: use $m \longrightarrow p \longrightarrow d$ as problem dictates

Ex: iid $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. (1) Does $\bar{X}_n \xrightarrow{d} \mu_x$
(2) Does $S_n^2 \xrightarrow{P} \sigma^2$ (consistency)

(1) $MSE[\bar{X}_n] \overset{V+B^2}{=} V[\bar{X}_n] + (E[\bar{X}_n] - \mu_x)^2$

$$\overset{iid}{=} \frac{\sigma_x^2}{n} + \overset{U.B.}{0} \qquad \therefore \downarrow 0 \text{ as } n \uparrow \infty$$

$\therefore \bar{X}_n \xrightarrow{m} \mu$ $\qquad$ $\therefore \bar{X}_n \xrightarrow{d} \mu$ by m-p-d

(2) Fact: $V[S_n^2] = \frac{2\sigma^4}{n-1}$ if iid $N(\mu, \sigma^2)$

earlier: S_n^2 u.B. for σ^2

$$\therefore MSE[S_n^2] \overset{V+B^2}{=} V[S_n^2] + (\underbrace{E[S_n^2] - \sigma^2}_{=0 \text{ u.B.}})^2$$

$$= \frac{2\sigma^4}{n-1} \quad \downarrow 0 \text{ as } n \uparrow \infty$$

$\therefore S_n^2 \xrightarrow{m} \sigma^2$ $\quad\therefore S_n^2 \xrightarrow{p} \sigma^2$ by m-p-d

$\therefore$ Consistent

Ex: Say X_n are similarly distributed as $X_n \sim N(0, \frac{1}{n^2})$
($\therefore$ not iid)

Define $Y_n \triangleq \sqrt{n} \cdot X_n$. Does $Y_1, Y_2, \ldots$ converge in probability to 0? Q: $Y_n \xrightarrow{p} 0$?

Method 1: $m \to p \to d$ approach ($m \to p$)

$Y_n \xrightarrow{m} 0$ iff $\lim_{n\to\infty} E[Y_n^2] = 0$

$$\therefore \lim_{n\to\infty} E[Y_n^2] = \lim_{n\to\infty} E[n \cdot X_n^2] = \lim_{n\to\infty} n \cdot E[X_n^2]$$

$$= \lim_{n\to\infty} n \cdot V[X_n] \quad \text{since } X_n \sim N(\underline{0}, \tfrac{1}{n^2})$$

$$= \lim_{n\to\infty} \frac{n}{n^2} \quad \text{since } X_n \sim N(0, \underline{\tfrac{1}{n^2}})$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore Y_n \xrightarrow{p} 0$ by $m \to p$ $\quad (m \to p \to d)$

Method 2: $MS = V + B^2$ and m-p-d.

since $X_n \sim N(0, \frac{1}{n^2})$

$$E[Y_n] = E[\sqrt{n} X_n] = \sqrt{n}\, E[X_n] = 0$$

$\therefore$ Y_n is U.B. for 0

since $X_n \sim N(0, \frac{1}{n^2})$

$$V[Y_n] = V[\sqrt{n} X_n] = n \cdot V[X_n] = \frac{n}{n^2} = \frac{1}{n}$$

$$\therefore \lim_{n\to\infty} MSE[Y_n] = \lim_{n\to\infty} \Big(\underbrace{V[Y_n]}_{=\frac{1}{n}} + \underbrace{(E[Y_n] - 0)^2}_{=0 \ (\text{U.B.})} \Big)$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore Y_n \xrightarrow{m} 0$ $\quad$ $\therefore Y_n \xrightarrow{P} 0$ by $m \to p$.

Method 3: Directly by definition of $Y_n \xrightarrow{P} Y$

pick $\varepsilon > 0$.

$$\therefore \lim_{n\to\infty} P[|Y_n - Y| > \varepsilon] = \lim_{n\to\infty} P[|\sqrt{n} X_n| > \varepsilon] = \lim_{n\to\infty} P[n X_n^2 > \varepsilon]$$

$$\overset{\text{M.I.}}{\leq} \lim_{n\to\infty} \frac{E[nX_n^2]}{\varepsilon^2} = \lim_{n\to\infty} \frac{n}{\varepsilon^2} E[X_n^2]$$

$$= \lim_{n\to\infty} \frac{n}{\varepsilon^2} \cdot V[X_n^2]$$ since $X_n \sim N(0, \frac{1}{n^2})$

$$= \lim_{n\to\infty} \frac{n}{n^2 \varepsilon^2}$$ since $X_n \sim N(0, \frac{1}{n^2})$

$$= 1/\varepsilon^2 \cdot \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore Y_n \xrightarrow{P} 0$

$\therefore V[S_n^2] < \infty$

Thrm: iid $X_1, X_2, \ldots$ & $\sigma_x^2 < \infty$ and $E[X^4] < \infty$

then $S_n^2 \xrightarrow{P} \sigma_x^2$. ($\therefore S_n^2$ is consistent for σ_x^2)

Prf:

$$S_n^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X}_n)^2$$

$$= \frac{1}{n-1} \sum_{k=1}^{n} (X_k^2 + \bar{X}_n^2 - 2 \cdot \bar{X}_n X_k)$$

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 + n \cdot \bar{X}_n^2 - 2 \cdot \bar{X}_n \sum_{k=1}^{n} X_k \right) \cdot \frac{n}{n}$$

$$= \frac{n}{n-1} \left(\frac{1}{n} \sum_{k=1}^{n} X_k^2 + X_n^2 - 2 \bar{X}_n^2 \right)$$

$$= \underline{\frac{n}{n-1}} \left(\frac{1}{n} \sum_{k=1}^{n} X_k^2 - \bar{X}_n^2 \right)$$

continuity theorem since $f(x) = x^2$ continuous.

$$\xrightarrow[\text{WLLN}]{P} 1 \cdot \left(E[X^2] - \mu_x^2 \right) = \sigma_x^2$$

QED.

Thrm: If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then $X_n + Y_n \xrightarrow{P} X + Y$.

Prf:

Lemma: If $a > 0$, $b > 0$, and $c > 0$ then
$a + b > c \longrightarrow \left(a > \frac{c}{2} \text{ or } b > \frac{c}{2} \right)$

Prf: Say not. $a + b > c$ & $\sim \left(a > \frac{c}{2} \text{ OR } b > \frac{c}{2} \right)$

$$\sim \left(a > \frac{c}{2} \text{ OR } b > \frac{c}{2} \right) \overset{\text{DeM}}{=} a \le \frac{c}{2},\ b \le \frac{c}{2}$$

$$\therefore a + b \le \frac{c}{2} + \frac{c}{2} = c$$

$\rightarrow\leftarrow$ contradiction

$\therefore a + b \le c$ and $a + b > c$.

Pick $\varepsilon > 0$:

$$\lim_{n\to\infty} P\left[\left|(X_n+Y_n)-(X+Y)\right| > \varepsilon\right]$$

$$= \lim_{n\to\infty} P\left[\left|(X_n-X)+(Y_n-Y)\right| > \varepsilon\right].$$

$$\leq \lim_{n\to\infty} P\left[|X_n-X|+|Y_n-Y| > \varepsilon\right]$$ since $|a+b| \leq |a|+|b|$ and P monotonicity

Lemma

$$\leq \lim_{n\to\infty} P\left[|X_n-X| > \tfrac{\varepsilon}{2} \text{ OR } |Y_n-Y| > \tfrac{\varepsilon}{2}\right]$$

$$\leq \underbrace{\lim_{n\to\infty} P\left[|X_n-X| > \tfrac{\varepsilon}{2}\right]}_{=0,\ \text{since } X_n \xrightarrow{P} X} + \underbrace{\lim_{n\to\infty} P\left[|Y_n-Y| > \tfrac{\varepsilon}{2}\right]}_{=0,\ \text{since } Y_n \xrightarrow{P} Y}.$$

$$= 0$$

$\therefore X_n + Y_n \xrightarrow{P} X+Y$

QED

Thrm: If $X_n \xrightarrow{o} X$ and $Y_n \xrightarrow{o} Y$ then $X_n + Y_n \xrightarrow{o} X+Y$

Prf:

Lemma: If $a_n \xrightarrow{e} a$ & $b_n \xrightarrow{e} b$ then $a_n + b_n \xrightarrow{e} a+b$.

Prf: By hypothesis for $\forall \varepsilon > 0$: $\exists n_x \in \mathbb{Z}^+$ & $\exists n_y \in \mathbb{Z}^+$:

$\forall n \geq n_x: |a_n - a| < \frac{\varepsilon}{2}$ $\qquad$ $\forall n \geq n_y: |b_n - b| < \frac{\varepsilon}{2}$.

$\therefore \forall n \geq \max(n_x, n_y)$,

$\therefore \left|(a_n+b_n)-(a+b)\right| = \left|(a_n-a)+(b_n-b)\right|$

$$\leq |a_n - a| + |b_n - b|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\therefore\ a_n + b_n \xrightarrow{c} a + b.$

QED (Lemma)

Let $A = \{\omega \in \Omega : X_n(\omega) \xrightarrow{P} X(\omega)\} \in \mathcal{A}$ (S.A.)

$B = \{\omega \in \Omega : Y_n(\omega) \xrightarrow{P} Y(\omega)\}$

$C = \{\omega \in \Omega : X_n(\omega) + Y_n(\omega) \xrightarrow{P} X(\omega) + Y(\omega)\}.$

by hypothesis: $P(A) = P(B) = 1.$

$$\therefore\ P(A^c) = P(B^c) = 0$$

Need to show: $P(C) = 1.$ Now: $P(C) = 1$ iff $P(C^c) = 0$

$\therefore$ Show $P(C^c) = 0$

Claim: $A \cap B \subset C$

Prf: pick $\omega \in A \cap B$ — Hypo

$\therefore\ \omega \in A$ and $\omega \in B$ — def $\cap$

$\therefore\ X_n(\omega) \xrightarrow{c} X(\omega)$ and $Y_n(\omega) \xrightarrow{c} Y(\omega)$ — defn A, B

$\therefore\ X_n(\omega) + Y_n(\omega) \xrightarrow{c} X(\omega) + Y(\omega)$ — Lemma

$\therefore\ \omega \in C$ — def C

$\therefore\ A \cap B \subset C$ — def $\subset$

QED (Claim)

contraposition, DeM

$\therefore C^c \subset (A \cap B)^c \overset{\text{DeM}}{=} A^c \cup B^c$

$\therefore P(C^c) \overset{\text{monotonocity, P}}{\leq} P(A^c \cup B^c) \overset{\text{Additivity (Boole)}}{\leq} P(A^c) + P(B^c) \overset{\text{by hypo}}{=} 0 + 0 = 0$

$\therefore P(C) = 1$

QED

Thrm: If $X_n \xrightarrow{m} X$ and $Y_n \xrightarrow{m} Y$ then $X_n + Y_n \xrightarrow{m} X + Y$.

Prf:

$$\lim_{n\to\infty} E\left[\left((X_n + Y_n) - (X+Y)\right)^2\right]$$

$$= \lim_{n\to\infty} E\left[\left((X_n - X) + (Y_n - Y)\right)^2\right]$$

$$= \lim_{n\to\infty} E\left[(X_n - X)^2\right] + \lim_{n\to\infty} E\left[(Y_n - Y)^2\right] + \lim_{n\to\infty} 2 \cdot E\left[(X_n - X)(Y_n - Y)\right]$$

$$= 0 + 0 + \lim_{n\to\infty} 2 \cdot \text{"}\sigma_{XY}\text{"} \quad \text{since } X_n \xrightarrow{m} X \text{ and } Y_n \xrightarrow{m} Y.$$

$$\leq 2 \cdot \lim_{n\to\infty} \text{"}\sigma_X\text{"} \cdot \text{"}\sigma_Y\text{"} \quad \text{by U.P. (Cauchy-Schwartz) } E[UV] \leq \sqrt{E[U^2] \cdot E[V^2]}$$

$$= 2 \cdot \lim_{n\to\infty} \sqrt{E\left[(X_n - X)^2\right]} \cdot \lim_{n\to\infty} \sqrt{E\left[(Y_n - Y)^2\right]}$$

$$= 2 \cdot \sqrt{\lim_{n\to\infty} E\left[(X_n - X)^2\right]} \cdot \sqrt{\lim_{n\to\infty} E\left[(Y_n - Y)^2\right]} \quad \text{since } f(x) = x^2 \text{ continuous.}$$

$$= 0 \quad \text{since } X_n \xrightarrow{m} X \text{ and } Y_n \xrightarrow{m} Y$$

$$\therefore X_n + Y_n \xrightarrow{m} X + Y$$

QED.

Caveat: $X_n + Y_n \xrightarrow{d} X + Y$ even if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$.

not in general

$p \longrightarrow d$ only for one sequence $X_1, X_2, X_3, \ldots$

in general